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LETTER TO THE EDITOR

On the higher-order generalizations of Poisson structures

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Abstract. The characterization of the Nambu–Poisson n-tensors as a subfamily of the generalized Poisson ones recently introduced (and here extended to the odd-order case) is discussed. The homology and cohomology complexes of both structures are compared, and some physical considerations are made.

1. Nambu–Poisson and generalized Poisson structures

a) Nambu–Poisson structures

The generalization of the Hamiltonian mechanics proposed by Nambu [1] more than 20 years ago has recently attracted renewed attention, particularly since Takhtajan [2] extended it further by introducing Poisson brackets (PB) involving an arbitrary number n of functions, the case $n = 3$ being Nambu's original proposal. His *Nambu–Poisson* (N–P) tensors provide an interesting generalization of the mathematical notion of *Poisson structure* (PS) on a manifold M [3]. A N–P structure is defined by an n-linear mapping $\{\cdot, \ldots, \cdot\} : \mathcal{F}(M) \times \overset{n}{\cdots} \times \mathcal{F}(M) \to \mathcal{F}(M)$ which is: (a) completely antisymmetric, (b) satisfies the Leibniz rule, i.e. $\{f_1, \ldots, f_{n-1}, gh\} = g\{f_1, \ldots, f_{n-1}, h\} + \{f_1, \ldots, f_{n-1}, g\}h$ and (c) verifies the (2n–1)-point, (n+1)-terms *fundamental identity* (FI) [2]

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} + \{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, \ldots, g_n\} + \cdots + \{g_1, \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\}. \tag{1}$$

This relation may be understood as expressing that the time evolution for $(n - 1)$ Hamiltonians $H_i, i = 1, \ldots, (n - 1)$ given by

$$\dot{f} = \{H_1, \ldots, H_{n-1}, f\} \tag{2}$$

is a derivation of the n-N–P bracket. The case $n = 3$ corresponds to Nambu's mechanics, although its associated five-point identity (equation (1) for $n = 3$), introduced by Sahoo and Valsakumar [4], was not explicitly mentioned in his work.

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The N–P bracket may be introduced through an antisymmetric contravariant tensor $\eta \in \wedge^n(M)$ or *multivector*, locally expressed by

$$\eta = \frac{1}{n!}\eta_{i_1\ldots i_n}\partial^{i_1}\wedge\cdots\wedge\partial^{i_n} \qquad \partial^i = \partial/\partial x^i \tag{3}$$

by defining

$$\{f_1,\ldots,f_n\} = \eta(\mathrm{d}f_1,\ldots,\mathrm{d}f_n). \tag{4}$$

Since (3) and (4) automatically guarantee properties (a) and (b) above, all that is required from η is to satisfy the FI. It is shown in [2] that this is achieved if the multivector η satisfies two conditions. The first is the 'differential condition'

$$\eta_{i_1\ldots i_{n-1}\rho}\partial^\rho\eta_{j_1\ldots j_n} - (\partial^\rho\eta_{i_1\ldots i_{n-1}j_1})\eta_{\rho j_2\ldots j_n} - (\partial^\rho\eta_{i_1\ldots i_{n-1}j_2})\eta_{j_1\rho j_3\ldots j_n} - \cdots$$
$$-(\partial^\rho\eta_{i_1\ldots i_{n-1}j_n})\eta_{j_1\ldots j_{n-1}\rho} = 0 \tag{5}$$

which we shall write here in compact form as

$$\eta_{i_1\ldots i_{n-1}\rho}\partial^\rho\eta_{j_1\ldots j_n} - \frac{1}{(n-1)!}\epsilon^{l_1\ldots l_n}_{j_1\ldots j_n}(\partial^\rho\eta_{i_1\ldots i_{n-1}l_1})\eta_{\rho l_2\ldots l_n} = 0. \tag{6}$$

The second condition, which follows from requiring that the terms with second derivatives of $f_1,\ldots,f_{n-1}$ in the FI should vanish, is the'algebraic condition'

$$\Sigma + P(\Sigma) = 0 \tag{7}$$

where Σ is the tensor of order $2n$ given by the sum of $(n+1)$ terms

$$\Sigma_{i_1\ldots i_n j_1\ldots j_n} = \eta_{i_1\ldots i_n}\eta_{j_1\ldots j_n} - \eta_{i_1\ldots i_{n-1}j_1}\eta_{i_n j_2\ldots j_n} - \eta_{i_1\ldots i_{n-1}j_2}\eta_{j_1 i_n j_3\ldots j_n}$$
$$-\eta_{i_1\ldots i_{n-1}j_3}\eta_{j_1 j_2 i_n j_4\ldots j_n} - \cdots - \eta_{i_1\ldots i_{n-1}j_n}\eta_{j_1 j_2\ldots j_{n-1}i_n} \tag{8}$$

and P interchanges the indices i_1 and j_1 in Σ†. Equation (8) may we rewritten as

$$\Sigma_{i_1\ldots i_n j_1\ldots j_n} = \frac{1}{n!}\epsilon^{l_1\ldots l_{n+1}}_{i_n j_1\ldots j_n}\eta_{i_1\ldots i_{n-1}l_1}\eta_{l_2\ldots l_{n+1}}. \tag{9}$$

Clearly, the algebraic condition is fulfilled if $\Sigma = 0$. This implies in turn that the skewsymmetric tensor η is decomposable (i.e. it can be written as an exterior product of vector fields on M) and in fact, as conjectured in [5], it may be shown [6–8] that all N–P tensors ($n > 2$) are decomposable (for $n = 2$, equation (7) is trivial).

b) Generalized Poisson structures

Recently, another generalization [9] of the ordinary PB has been proposed under the name of *generalized PS* (GPS) by extending the geometrical approach to standard PS [3]. In these, a bivector $\Lambda \in \wedge^2(M)$ on a manifold M defines a PS *iff* it has a vanishing Schouten–Nijenhuis bracket (SNB) with itself, $[\Lambda,\Lambda] = 0$. This condition, when generalized to multivectors of even order $\Lambda \in \wedge^{2p}(M)$ provides the definition of the GPS (see below for the odd-order case) because for

$$\Lambda = \frac{1}{(2p)!}\omega_{j_1\ldots j_{2p}}\,\partial^{j_1}\wedge\cdots\wedge\partial^{j_{2p}} \tag{10}$$

† From the condition $\Sigma = 0$ easily follows that in a n-dimensional space the (obviously decomposable) n-tensor $\eta_{i_1\ldots i_n} = \epsilon_{i_1\ldots i_n}$ defining the $\mathbb{R}^n$ volume element and the tensor $\eta_{i_1\ldots i_{n-1}}(x) = \epsilon_{i_1\ldots i_n}x^{i_n}$ are Nambu tensors [5], i.e. satisfy the conditions (6) and (7).

the requirement $[\Lambda, \Lambda] = 0$ means that the coordinates of the *generalized Poisson* (GP) multivector Λ satisfy the condition [9]

$$\epsilon^{j_1\dots j_{4p-1}}_{i_1\dots i_{4p-1}} \omega_{j_1 j_2 \dots j_{2p-1} k} \partial^k \omega_{j_{2p}\dots j_{4p-1}} = 0 \tag{11}$$

which is equivalent to the $(4p-1)$-point, $\binom{4p-1}{2p-1}$-terms *generalized Jacobi identity* (GJI)

$$\epsilon^{j_1\dots j_{4p-1}}_{1\dots 4p-1} \{f_{j_1}, f_{j_2}, \dots, f_{j_{2p-1}}, \{f_{j_{2p}}, \dots, f_{j_{4p-1}}\}\} = 0 \tag{12}$$

where the *generalized Poisson bracket* (GPB) is also defined by (4) but for the Λ in (10). Notice that, as we shall see below, no further conditions are needed to remove the second derivatives from equation (12), which is already free of them. As a result the $2p$-vector is constrained by the differential condition (11) *only*.

The even GPS have a clear differential geometrical origin due to their definition in terms of the SNB by the condition $[\Lambda, \Lambda] = 0$. Moreover, in the linear case one can find (an infinite number of) examples of even GPS defined by the Lie algebra cohomology cocycles [9]. Indeed, for simple Lie algebras of rank l, there are l antisymmetric tensors provided by the l $(2p_i - 1)$-cocycles $(i = 1, \dots, l)$ [10] with coordinates $\Omega_{j_1\dots j_{2p_i-2}}{}^{\sigma}$, which define GP tensors of order $(2p_i - 2)$,

$$\omega_{j_1\dots j_{2p-2}} = \Omega_{j_1\dots j_{2p-2}}{}^{\sigma} x_{\sigma} \tag{13}$$

which satisfy (11). These linear GPB may be seen to be the analogues of the even multibrackets defining higher order Lie algebras [11] and, from this point of view, there is a one-to-one correspondence between these linear GPB and the higher-order brackets of associative non-commuting operators. The time evolution, defined as in (2) but for $(2p-1)$ Hamiltonians, is not a derivation of the GPB as it is in the N–P structure. In contrast with the N–P tensors, however, the GP $2p$-multivectors (10) are not decomposable in general because they do not need to obey the algebraic condition (7). It is easy to check, on the other hand, that any decomposable multivector of order $2p$, $p > 1$, defines a GPS. Indeed, in this case $\Lambda = X_1 \wedge \cdots \wedge X_{2p}$ and using standard properties of the SNB [9, equation (4.1) of second reference] it follows that

$$\begin{aligned}[\Lambda, \Lambda] &= [X_1 \wedge \cdots \wedge X_{2p}, X_1 \wedge \cdots \wedge X_{2p}] \\ &= \sum (-1)^{t+s} X_1 \wedge \cdots \hat{X}_s \cdots \wedge X_{2p}, \wedge [X_s, X_t] \wedge X_1 \wedge \cdots \hat{X}_t \cdots \wedge X_{2p} = 0\end{aligned} \tag{14}$$

due to the appearance of repeated vector fields.

Much in the same way that on a Poisson manifold it is possible to define a Poisson cohomology [3], a GPB also defines a *generalized Poisson cohomology* [9] through the SNB. Explicitly, if the $2p$-vector Λ defines a GPS, the mapping $\delta_\Lambda : \wedge^q(M) \to \wedge^{2p+q-1}(M)$ defined by

$$\delta_\Lambda : \alpha \mapsto [\Lambda, \alpha] \tag{15}$$

is nilpotent since $[\Lambda, [\Lambda, \alpha]] = 0$ and defines a $(2p-1)$-degree cohomology operator.

Equation (14) and the decomposability of all N–P tensors show that there is an overlap among the above generalizations of the standard PS. This may be shown directly by noticing first that the GJI (12) is a full antisymmetrization of (1) †. This observation leads to the following simple lemma.

Lemma 1. A N–P bracket (hence, satisfying the FI (1)) verifies

$$\epsilon^{j_1\dots j_{2n-1}}_{1\dots 2n-1} \{f_{j_1}, f_{j_2}, \dots, f_{j_{n-1}}, \{f_{j_n}, \dots, f_{j_{2n-1}}\}\} = 0. \tag{16}$$

† This fact was also known to L Takhtajan (private communication).

Proof. Multiplying both sides of (1) by ϵ and using its antisymmetry, (1) is rewritten as

$$\epsilon^{j_1\ldots j_{2n-1}}_{1\ldots 2n-1}\{f_{j_1}, f_{j_2}, \ldots, f_{j_{n-1}}, \{f_{j_n}, \ldots, f_{j_{2n-1}}\}\}$$
$$= n(-1)^{n-1}\epsilon^{j_1\ldots j_{2n-1}}_{1\ldots 2n-1}\{f_{j_1}, f_{j_2}, \ldots, f_{j_{n-1}}, \{f_{j_n}, \ldots, f_{j_{2n-1}}\}\} \tag{17}$$

hence, for $n \geqslant 2$, we obtain (16), QED (for $n = 2$ the N–P and the GPS reduce to the standard PS).

Equation (16), for $n = 2p$, is the same as (12) and we conclude that *every N–P bracket of even order also defines a generalized Poisson bracket* [12]. □

Due to the geometrical origin of the GJI condition, the GPS were originally introduced [9] for even order only: the SNB of a p (q)-multivector A (B) satisfies $[A, B] = (-1)^{pq}[B, A]$ and thus $[\Lambda, \Lambda] \equiv 0$ if Λ is of odd order (we are not including here the case of the 'super' SNB [13]). Nevertheless, we may extend the GPS by adopting the GJI (16) for arbitrary (even or odd) n as a first step in their definition. In the odd case, the GJI is unrelated to the condition $[\Lambda, \Lambda] = 0$ since it is trivially satisfied. But if we now define an odd-order GPB satisfying (16) for *n odd*, we find setting $f_i = x_i$, $i = 1, \ldots, 2n-1$ that the coordinates of the associated n-vector Λ must satisfy the differential condition (cf (11), (6))

$$\epsilon^{j_1\ldots j_{2n-1}}_{i_1\ldots i_{2n-1}}\omega_{j_1 j_2\ldots j_{n-1}k}\partial^k\omega_{j_n\ldots j_{2n-1}} = 0. \tag{18}$$

For n odd a second step now becomes necessary to cancel all second derivatives that appear in the GJI (16). If we want to keep the GJI for odd-order brackets we have to impose an additional 'algebraic condition' to the n vector defining the structure. Explicitly, this condition (for arbitrary n) is (cf (7))

$$\epsilon^{i_1\ldots i_{n-1}\, j_1\ldots j_{n-1}}_{k_1\ldots\ldots\ldots k_{2n-2}}(\omega_{i_1\ldots i_{n-1}\rho}\omega_{j_1\ldots j_{n-1}\sigma} + \omega_{i_1\ldots i_{n-1}\sigma}\omega_{j_1\ldots j_{n-1}\rho}) = 0. \tag{19}$$

For even n this equation is automatically satisfied; this explains why there is no 'algebraic condition' for even multivectors defining a GPS. In contrast, equation (19) is an additional condition on ω for n odd.

As a consequence of lemma 1, conditions (18) and (19) must be extracted from conditions (6) and (7). In fact, it is easily deduced that (18) follows (only) from (6) and that (19) comes (only) from (7).

Summarizing, extending the definition of GPS to odd brackets, the following general lemma follows.

Lemma 2. The N–P tensors of even or odd order are a subclass of the multivectors defining the GPS, namely those for which the time evolution is a derivation of the bracket (or, in other words, the time-evolution operator preserves the Poisson n-bracket structure).

We conclude this section by mentioning that one might think of using Lie algebra cocycles $\Omega_{i_1\ldots i_{2p}\sigma}$ as the coordinates of a $(2p+1)$-vector Λ leading to the odd bracket $\{f_{i_1}, \ldots, f_{i_{2p}}, f_\sigma\} = \Lambda(\mathrm{d}f_{i_1}, \ldots, \mathrm{d}f_{i_{2p}}, \mathrm{d}f_\sigma)$ (see [14] for the trilinear case; cf [15]). However, although the differential condition for both the N–P (equation (6)) and odd GPS (equation (18)) are trivially satisfied for a constant multivector, this is not in general the case for the algebraic N–P (equation (7)) and odd GPS (equation (19)) conditions. In contrast, any cocycle defines an even linear GPS.

2. Homology and cohomology

We now compare the homological complexes underlying both structures (N–P, (a) and GPS, (b)). First, let us recall the standard homology complex for a Lie algebra $\mathcal{G}$. The n-chains are

n-vectors of $\wedge^n(\mathcal{G})$ (for instance, left-invariant [LI] n-antisymmetric contravariant tensors on the associated group G, i.e. LI elements of $\wedge^n(T(G))$), and the homology operator $\partial C_n \to C_{n-1}$ is defined by

$$\partial(x_1 \wedge \cdots \wedge x_n) = \sum_{1\leqslant l<k\leqslant n} (-1)^{l+k+1}[x_l, x_k] \wedge x_1 \wedge \cdots \hat{x}_l \cdots \hat{x}_k \cdots \wedge x_n \tag{20}$$

where $x \in \mathcal{G}$ and [,] is the Lie bracket in $\mathcal{G}$; $\partial[\wedge^n(\mathcal{G})] = 0$ for $n \leqslant 1$. In particular, $\partial(x_1 \wedge x_2) = [x_1, x_2]$ and, in this case, ∂ may be relabelled $\partial \equiv \partial_2$, $\partial_2 : \wedge^n(\mathcal{G}) \to \wedge^{n-(2-1)}(\mathcal{G})$.

a1) Nambu–Lie homology

Let us consider now a *Nambu–Lie* (N–L) *algebra* $\mathcal{V}$ of order s in the sense of [16]†. This means that there is an antisymmetric s-bracket $[\cdot, \overset{s}{\ldots}, \cdot] : \mathcal{V} \times \overset{s}{\cdots} \times \mathcal{V} \to \mathcal{V}$, $[x_1, \ldots, x_s] \in \mathcal{V}$ which satisfies the FI

$$\begin{aligned}[x_1, \ldots, x_{s-1}, [y_1, \ldots, y_s]] = [[x_1, \ldots, x_{s-1}, y_1], y_2, \ldots, y_s] \\ +[y_1, [x_1, \ldots, x_{s-1}, y_2], \ldots, y_s] + \cdots + [y_1, \ldots, y_{s-1}, [x_1, \ldots, x_{s-1}, y_n]]\end{aligned} \tag{21}$$

i.e. such that the map $[x_1, \ldots, x_{s-1}, \cdot] : \mathcal{V} \to \mathcal{V}$ is a multiderivation of the N–L bracket. The N–L homology has been introduced by Takhtajan [16]. Let C_n be the n-chains $C_n = \mathcal{V} \otimes \overset{n(s-1)+1}{\cdots} \otimes \mathcal{V}$, $C_0 = \mathcal{V}$. It is convenient to denote the arguments in the chains C_n by

$$(X_1, X_2, \ldots, X_n, x) = (x_{i_1^1}, \ldots, x_{i_{s-1}^1}, x_{i_1^2}, \ldots, x_{i_{s-1}^2}, \ldots, x_{i_1^n}, \ldots, x_{i_{s-1}^n}, x) \tag{22}$$

where $X_1 = (x_{i_1^1}, \ldots, x_{i_{s-1}^1}) \in \mathcal{V}^{s-1}$, etc and $x \in \mathcal{V}$. Now we consider the dot products $C_1 \times C_1 \to C_1$ and $C_1 \times \mathcal{V} \to \mathcal{V}$ defined by

$$X \cdot Y := \sum_{i=1}^{n-1} y_1 \otimes \cdots \otimes [x_1, \ldots, x_{n-1}, y_i] \otimes \cdots y_{n-1} \tag{23}$$

$$X \cdot x := [x_1, \ldots, x_{n-1}, x]. \tag{24}$$

Due to the FI (equation (21)) these products satisfy

$$X \cdot (Y \cdot Z) - (X \cdot Y) \cdot Z = Y \cdot (X \cdot Z) \qquad X \cdot (Y \cdot z) - (X \cdot Y) \cdot z = Y \cdot (X \cdot z). \tag{25}$$

If these products were antisymmetric (25) would be the Jacobi identity and thus, we would have defined a Lie algebra. Although they are not, we can still define a Lie-type homology because the operator ∂_s defined on C_1 by $\partial_s : C_1 \to C_0 = \mathcal{V}$, $\partial_s : (x_1, \ldots, x_s) \mapsto [x_1, \ldots, x_s]$ and on C_n by

$$\begin{aligned}\partial_s(X_1, \ldots, X_n, x) = \sum_{1\leqslant i<j\leqslant n} (-1)^{i+1}(X_1, \ldots, \hat{X}_i, \ldots, X_i \cdot X_j, \ldots, X_n, x) \\ + \sum_{1\leqslant i\leqslant n} (-1)^{i+1}(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}, X_i \cdot x)\end{aligned} \tag{26}$$

verifies‡ $\partial_s^2 = 0$. On two-chains, $\partial_s^2 = 0$ gives the 'fundamental identity' which replaces the Jacobi identity for N–L algebras. For instance, for $s = 4$ we have $\partial_4(x_1, x_2, x_3, x_4) =$

† The case of the more general Nambu–Leibniz s-algebra (which does not assume the antisymmetry of the bracket [17]) is discussed in [18]. We thank L Takhtajan for sending this paper to us.

‡ This is the case for the Leibniz algebras [17] where we have a Lie-like homology in which the 'bracket' is not antisymmetric. The Jacobi-like conditions (25) ensure that ∂_s is nilpotent.

$[x_1, x_2, x_3, x_4]$ and ∂_4^2 on C_2 gives (cf (21))

$$\partial^2(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = [[x_1, x_2, x_3, x_4], x_5, x_6, x_7] + [x_4, [x_1, x_2, x_3, x_5], x_6, x_7] \\ +[x_4, x_5, [x_1, x_2, x_3, x_6], x_7] + [x_4, x_5, x_6, [x_1, x_2, x_3, x_7]] \\ -[x_1, x_2, x_3, [x_4, x_5, x_6, x_7]]. \qquad (27)$$

b1) GP–Lie homology

Let us now look at the case of even GPS. To this end, consider a *higher-order Lie algebra* in the sense of [11] (see also [19, 20]), i.e. let $\mathcal{G}$ be a vector space endowed with an antisymmetric s-linear operation (s even) $[\cdot, \overset{s}{\ldots}, \cdot] : \mathcal{G} \otimes \overset{s}{\cdots} \otimes \mathcal{G} \to \mathcal{G}$, which verifies the generalized Jacobi identity

$$\frac{1}{s!}\frac{1}{(s-1)!}\sum_{\sigma\in S_{2s-1}}(-1)^{\pi(\sigma)}[[x_{\sigma(1)}, \ldots, x_{\sigma(s)}], x_{\sigma(s+1)}, \ldots, x_{\sigma(2s-1)}] = 0. \qquad (28)$$

In particular, if s is even the s-bracket of associative operators defined by

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_s}] = \sum_{\sigma\in S_s}(-1)^{\pi(\sigma)} x_{i_{\sigma(1)}} x_{i_{\sigma(2)}} \ldots x_{i_{\sigma(s)}} \qquad (29)$$

satisfies (28) (for s odd, the l.h.s. in (28) is proportional to $[x_1, \ldots, x_{2s-1}]$ rather than zero [11]).

The n-chains are now elements of $\wedge^n(\mathcal{G})$ and the homology operator ∂_s is the linear mapping $\partial_s : \wedge^n(\mathcal{G}) \to \wedge^{n-(s-1)}(\mathcal{G})$ defined by

$$\partial_s(x_1 \wedge \cdots \wedge x_n) = \frac{1}{s!(n-s)!}\epsilon^{i_1 \ldots i_n}_{1 \ldots n}\partial_s(x_{i_1} \wedge \cdots \wedge x_{i_s}) \wedge x_{i_{s+1}} \wedge \cdots \wedge x_{i_n}. \qquad (30)$$

Denoting, $\partial_s(x_{i_1}, \ldots, x_{i_s}) = [x_{i_1}, \ldots, x_{i_s}] \in \wedge(\mathcal{G})$ equation (30) may be rewritten

$$\partial_s(x_1 \wedge \cdots \wedge x_n) = \sum_{1\leqslant i_1<\cdots<i_s\leqslant n}(-1)^{i_1+\cdots+i_s+s/2} \\ \times[x_{i_1}, \ldots, x_{i_s}] \wedge x_1 \wedge \cdots \wedge \hat{x}_{i_1} \wedge \cdots \wedge \hat{x}_{i_s} \wedge \cdots \wedge x_n \qquad (31)$$

and the GJI may be also expressed as $\partial_s^2[\wedge^{2s-1}(\mathcal{G})] = 0$. For instance, for $s = 4$, $\partial_4^2(x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \wedge x_{i_4} \wedge x_{i_5} \wedge x_{i_7})$ gives the GJI (equation (28)) which is the sum of

$7!/4!3! = 35$ terms

$$
\begin{aligned}
&[[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}], x_{i_5}, x_{i_6}, x_{i_7}] - [[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_5}], x_{i_4}, x_{i_6}, x_{i_7}] \\
&\quad + [[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_6}], x_{i_4}, x_{i_5}, x_{i_7}] - [[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_7}], x_{i_4}, x_{i_5}, x_{i_6}] \\
&\quad + [[x_{i_1}, x_{i_2}, x_{i_4}, x_{i_5}], x_{i_3}, x_{i_6}, x_{i_7}] - [[x_{i_1}, x_{i_2}, x_{i_4}, x_{i_6}], x_{i_3}, x_{i_5}, x_{i_7}] \\
&\quad + [[x_{i_1}, x_{i_2}, x_{i_4}, x_{i_7}], x_{i_3}, x_{i_5}, x_{i_6}] + [[x_{i_1}, x_{i_2}, x_{i_5}, x_{i_6}], x_{i_3}, x_{i_4}, x_{i_7}] \\
&\quad - [[x_{i_1}, x_{i_2}, x_{i_5}, x_{i_7}], x_{i_3}, x_{i_4}, x_{i_6}] + [[x_{i_1}, x_{i_2}, x_{i_6}, x_{i_7}], x_{i_3}, x_{i_3}, x_{i_7}] \\
&\quad - [[x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}], x_{i_2}, x_{i_6}, x_{i_7}] + [[x_{i_1}, x_{i_3}, x_{i_4}, x_{i_6}], x_{i_2}, x_{i_5}, x_{i_7}] \\
&\quad - [[x_{i_1}, x_{i_3}, x_{i_4}, x_{i_7}], x_{i_2}, x_{i_5}, x_{i_6}] - [[x_{i_1}, x_{i_3}, x_{i_5}, x_{i_6}], x_{i_2}, x_{i_4}, x_{i_7}] \\
&\quad + [[x_{i_1}, x_{i_3}, x_{i_5}, x_{i_7}], x_{i_2}, x_{i_4}, x_{i_6}] - [[x_{i_1}, x_{i_3}, x_{i_6}, x_{i_7}], x_{i_2}, x_{i_4}, x_{i_5}] \\
&\quad + [[x_{i_1}, x_{i_4}, x_{i_5}, x_{i_6}], x_{i_2}, x_{i_3}, x_{i_7}] - [[x_{i_1}, x_{i_4}, x_{i_5}, x_{i_7}], x_{i_2}, x_{i_3}, x_{i_6}] \\
&\quad + [[x_{i_1}, x_{i_4}, x_{i_6}, x_{i_7}], x_{i_2}, x_{i_3}, x_{i_5}] - [[x_{i_1}, x_{i_5}, x_{i_6}, x_{i_7}], x_{i_2}, x_{i_3}, x_{i_4}] \\
&\quad + [[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}], x_{i_1}, x_{i_6}, x_{i_7}] - [[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_6}], x_{i_1}, x_{i_5}, x_{i_7}] \\
&\quad + [[x_{i_2}, x_{i_3}, x_{i_4}, x_{i_7}], x_{i_1}, x_{i_5}, x_{i_6}] + [[x_{i_2}, x_{i_3}, x_{i_5}, x_{i_6}], x_{i_1}, x_{i_4}, x_{i_7}] \\
&\quad - [[x_{i_2}, x_{i_3}, x_{i_5}, x_{i_7}], x_{i_1}, x_{i_4}, x_{i_6}] + [[x_{i_2}, x_{i_3}, x_{i_6}, x_{i_7}], x_{i_1}, x_{i_4}, x_{i_5}] \\
&\quad - [[x_{i_2}, x_{i_4}, x_{i_5}, x_{i_6}], x_{i_1}, x_{i_3}, x_{i_7}] + [[x_{i_2}, x_{i_4}, x_{i_5}, x_{i_7}], x_{i_1}, x_{i_3}, x_{i_6}] \\
&\quad - [[x_{i_2}, x_{i_4}, x_{i_6}, x_{i_7}], x_{i_1}, x_{i_3}, x_{i_5}] + [[x_{i_2}, x_{i_5}, x_{i_6}, x_{i_7}], x_{i_1}, x_{i_3}, x_{i_4}] \\
&\quad + [[x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}], x_{i_1}, x_{i_2}, x_{i_7}] - [[x_{i_3}, x_{i_4}, x_{i_5}, x_{i_7}], x_{i_1}, x_{i_2}, x_{i_6}] \\
&\quad + [[x_{i_3}, x_{i_4}, x_{i_6}, x_{i_7}], x_{i_1}, x_{i_2}, x_{i_5}] - [[x_{i_3}, x_{i_5}, x_{i_6}, x_{i_7}], x_{i_1}, x_{i_2}, x_{i_4}] \\
&\quad + [[x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}], x_{i_1}, x_{i_2}, x_{i_3}] = 0.
\end{aligned}
\tag{32}
$$

For the even linear GPS constructed from odd Lie algebra cocycles, the above GJI truly reflect the underlying Lie-algebra structure; this justifies the GP–*Lie* name given to this case. These GJI are particular examples of those appearing in the strongly homotopy algebras [21], which contain 'controlled' violations of the above GJI which may be introduced in our scheme by using a suitable modification of the *complete BRST operator* associated to $\mathcal{G}$ [11, theorem 5.2]. These algebraic structures have been found relevant in closed-string field theory (see [21, 11] and references therein).

a2) Nambu–Lie cohomology

Let us now consider the dual cohomology operations. For the N–L case we define n-cochains C^n as mappings $\alpha : \mathcal{V} \otimes \overset{n(s-1)+1}{\cdots} \otimes \mathcal{V} \to \mathcal{A}$ where $\mathcal{A}$ is an Abelian algebra (real field, for instance). Then, the cohomology operator $\delta_s : C^n \to C^{n+1}$ is defined as the dual of the homology operator ∂_s, $(C^n, \partial_s C_{n+1}) = (\delta_s C^n, C_{n+1})$ where (,) denotes the natural pairing between chains and cochains. Using this duality it follows immediately that the operator δ_s is defined (cf [7]) by its action on a cochain $\alpha \in C^p$ by

$$
\begin{aligned}
(\delta_s\alpha)(X_1, \ldots, X_{p+1}, x) = &\sum_{1 \leqslant i < j \leqslant p+1} (-1)^{i+1} \alpha(X_1, \ldots, \hat{X}_i, \ldots, X_i \cdot X_j, \ldots X_{p+1}, x) \\
&+ \sum_{1 \leqslant i \leqslant p+1} (-1)^{i+1} \alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}, X_i \cdot x)
\end{aligned}
\tag{33}
$$

whereas in the homology case $X = (x_1, \ldots, x_{s-1}) \in \mathcal{V}^{s-1}$ and $x \in \mathcal{V}$. The proof that $\delta_s^2 = 0$ is analogous to that for the Lie algebra cohomology coboundary operator if one thinks of $X_i \cdot X_j$ in (33) as a commutator, in which case equation (25) looks like a Jacobi identity.

b2) GP–Lie cohomology

In the case of the linear GPS constructed on the dual of a Lie algebra we may introduce a cohomology operator dual to the homology one given in equation (31). Acting on n-cochains $\alpha_{i_1\ldots i_n}$

$$(\delta_s\alpha)(x_1,\ldots,x_{n+s-1}) = \sum_{1\leqslant i_1<\cdots<i_s\leqslant n+s-1} (-1)^{i_1+\cdots+i_s+s/2} \times\alpha([x_{i_1},\ldots,x_{i_s}],x_1,\ldots,\hat{x}_{i_1},\ldots,\hat{x}_{i_s},\ldots,x_{n+s-1}) \tag{34}$$

or equivalently, setting $[x_{i_1},\ldots,x_{i_s}] = \omega_{i_1\ldots i_s}{}^{\rho}x_\rho$ for definiteness,

$$(\delta_s\alpha)_{i_1\ldots i_{n+s-1}} = \frac{1}{s!(n-1)!}\epsilon^{j_1\ldots j_{n+s-1}}_{i_1\ldots i_{n+s-1}}\omega_{j_1\ldots j_s}{}^{\rho}\alpha_{\rho j_{s+1}\ldots j_{n+s-1}}. \tag{35}$$

The nilpotency of δ_s follows from checking that [9]

$$(\delta_s^2\alpha)_{i_1\ldots i_{2s+n-2}} = \frac{s}{(s!)^2(n-1)!}\epsilon^{j_1\ldots j_s k_1\ldots k_{s+n-2}}_{i_1\ldots i_{2s+n-2}}\omega_{j_1\ldots j_s}{}^{\rho}\omega_{\rho k_1\ldots k_{s-1}}{}^{\sigma}\alpha_{\sigma k_s\ldots k_{s+n-2}} + \frac{(n-1)}{(s!)^2(n-1)!}\epsilon^{j_1\ldots j_s k_1\ldots k_{s+n-2}}_{i_1\ldots i_{2s+n-2}}\omega_{j_1\ldots j_s}{}^{\rho}\omega_{k_1\ldots k_s}{}^{\sigma}\alpha_{\sigma\rho k_{s+1}\ldots k_{s+n-2}} = 0 \tag{36}$$

where the second term is zero since s is even and the cochain α is antisymmetric in (ρ,σ) and the first one is also zero since it encompasses the GJI. Since this cohomology is based on multi-algebra commutators, it applies to *linear* GPS. For a general GPS, however, the operator (35) is not defined, but the associated $2p$-vector Λ still defines a generalized Poisson cohomology by (15).

3. Concluding (physical) remarks

The n-dimensional phase space of Nambu [1] for the N–P structure associated with the volume element in $\mathbb{R}^n$, determined by an n-vector x_i, has a divergenceless velocity field since, by (2),

$$\partial^j\dot{x}_j = \partial^j\{H_1,\ldots,H_{n-1},x_j\} = \partial^j\epsilon_{i_1\ldots i_{n-1}j}\frac{\partial H_1}{\partial x_{i_1}}\cdots\frac{\partial H_{n-1}}{\partial x_{i_{n-1}}} = 0. \tag{37}$$

This analogue of the *Liouville theorem* (a main motivation in Nambu's generalization of Hamiltonian dynamics) also holds for the linear GPS given by the cocycles (13) since $\omega_{i_1\ldots i_{2m-2}} = x_\sigma\Omega_{i_1\ldots i_{2m-2}\cdot}{}^{\sigma}$ and $\Omega_{i_1\ldots i_{2m-2}\cdot}{}^{\sigma}$ is a constant antisymmetric $(2m-1)$-tensor. Thus,

$$\partial^j\dot{x}_j = \partial^j(\omega_{i_1\ldots i_{2m-3}j}\partial^{i_1}H_1\ldots\partial^{i_{2m-3}}H_{2m-3}) = \partial^j(x_\sigma\Omega_{i_1\ldots i_{2m-3}j\cdot}{}^{\sigma})\partial^{i_1}H_1\ldots\partial^{i_{2m-3}}H_{2m-3} = \Omega_{i_1\ldots i_{2m-3}j\cdot}{}^{j} = 0. \tag{38}$$

More generally, the conservation equation is clearly satisfied when the GPS on M is defined by $\omega_{i_1\ldots i_{2m-2}} = \partial^l\tilde{\omega}_{li_1\ldots i_{2m-2}}$, and $\tilde{\omega}$ is an odd-order antisymmetric tensor†.

The Poisson theorem states that the PB of two integrals of motion is also an integral of motion. In N–P mechanics the extension of the Poisson theorem is guaranteed by the FI [2] that may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\{g_1,\ldots,g_n\} = \sum_{i=1}^{n}\{g_1,\ldots,\frac{\mathrm{d}g_i}{\mathrm{d}t},\ldots,g_n\}. \tag{39}$$

† The previous case of the linear GPS is included here because one may take $\tilde{\omega}_{li_1\ldots i_{2m-2}} = \frac{1}{(2m-2)!}\frac{1}{n-2m+3}\epsilon^{jj_1\ldots j_{2m-2}}_{li_1\ldots i_{2m-2}}x_j x_\sigma\Omega^{\sigma}{}_{j_1\ldots j_{2m-2}}$, $i,j,l = 1,\ldots,n$ where n is the dimension of M. Note that the second denominator in the last expression cannot vanish because the order of the bracket $(2m-2)$ never exceeds the dimension n of the manifold.

For the GPS, there is also an analogue of the Poisson theorem, although the condition required for the constants of the motion $(g_1, \ldots, g_k)$ $(k \geqslant 2p)$ is more stringent. Not only the g's have to be constants of the motion, $\{g_i, H_1, \ldots, H_{2p-1}\} = 0$, $i = 1, \ldots, k$: the set $(g_{i_1}, \ldots, g_{i_{2p-1}}, H_1, \ldots, H_{2p-1})$ of any $(2p-1)$ constants of the motion and the $(2p-1)$ Hamiltonians has to be in involution, i.e. any subset of $2p$ elements has to have zero GPB [9, theorem 6.1]. This is because, in contrast with (1), where the $f_1, \ldots f_{n-1}$ may play the role of Hamiltonians, the GJI in (12) includes GPB which contain Hamiltonians and more than one constant of the motion.

We would like to conclude with a comment concerning quantization. As pointed out by Nambu himself [1], the antisymmetry property is necessary to have Hamiltonians that are constants of the motion in Nambu's mechanics. This also applies to the higher-order N–P structures [2], and remains true as well of the GPS in [9]. The structure of the FI makes the N–P bracket in [2] specially suitable for the differential equation describing the time evolution of a dynamical quantity. Nevertheless, the standard quantization of Nambu mechanics is an open problem likely without solution (see [15] in this respect). There is a simple argument against an elementary quantization of N–P mechanics in which one tries to keep the standard one-to-one correspondence among certain dynamical quantities, their associated quantum operators and the infinitesimal generators of the invariance groups. It is physically natural to assume that these quantum operators, x_i say, are associative. But if so, it is not difficult to check [11] that any commutator $[x_1, \ldots, x_s]$ defined by the antisymmetrized sum of their products, as in (29), does not satisfy the FI. For odd s-brackets, moreover, we find [11, lemma 2.1] that it is not possible to realize the GJI in terms of odd multibrackets, since then the r.h.s. of equation (28) is replaced by $[x_1, \ldots, x_{2s-1}]$. Thus, for the odd case (which includes Nambu's) a multibracket of associative operators defined as in (29) leads to an identity which is *outside* the original N–P algebraic structure. For s even, however, equation (28) holds. The resulting identity, however, is not the FI, but the GJI associated with the GPS introduced in [9]. Thus, a natural correspondence between multibrackets and higher-order PB exists only for the even multibrackets and the GPS. The associativity of the quantum operators is not compatible with the derivation property of the N–P bracket which leads to the FI (1). Such a compatibility exists for the even GPS; however, in this case the time evolution fails to be a derivation of the GPB making it more difficult to establish a dynamics already at the classical level.

The above discussion indicates that, in Nambu's words, 'quantum theory is pretty unique although its classical analog may not be'. The quantization of higher-order Poisson brackets requires renouncing to some of the standard steps towards quantum mechanics†. But it may well be (see also [24]) that classical *mechanics* is pretty unique too if the term 'dynamical system' is restricted to its physical (rather than mathematical) meaning.

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† A formal quantization of the n=3 Nambu mechanics case, where the Nambu bracket itself is replaced by an $\hbar$-deformed one, has been performed in [22]. For the 'quantization deformation' and $*$ products, see the references quoted in [22] and also [23].

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